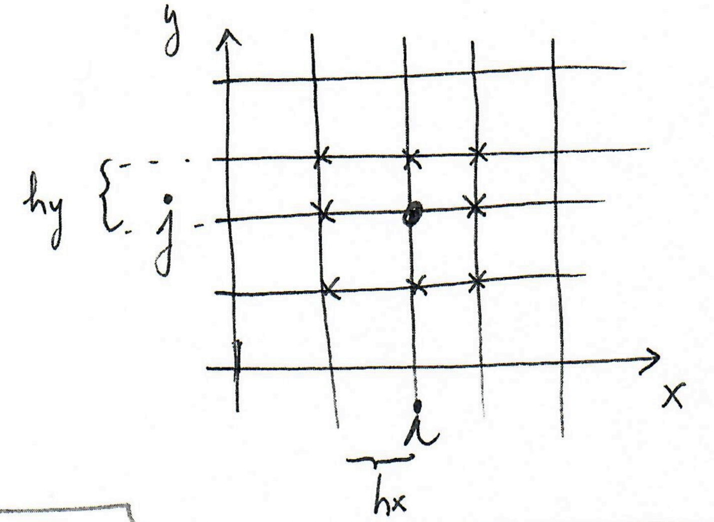
# 3.5 The 9-point Laplacian.

Again, we want to approximate

$$\boxed{\nabla^2 u = 0} \qquad (0.1)$$

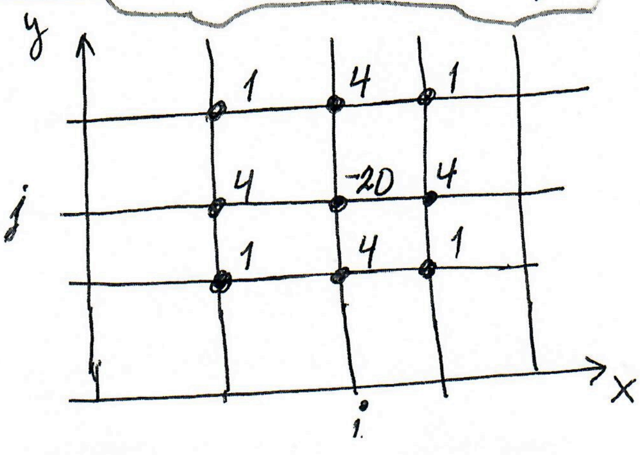
But, this time we will use a 9-point stencil as




$h_x = h_y = h$.

$h = \frac{1}{m+1}$

$$\boxed{\nabla_9^2 U_{ij} = \frac{1}{6h^2}\Big[U_{i-1,j-1} + 4U_{i,j-1} + U_{i+1,j-1} + 4U_{i-1,j} - 20U_{ij} + 4U_{i+1,j} + U_{i-1,j+1} + 4U_{i,j+1} + U_{i+1,j+1}\Big] \equiv \nabla_9^2 U_{i,j}} \qquad (1.1)$$



Substitution of the exact solution $u_{ij} = u(x_i, y_j)$ into (1.1) leads to

$$\nabla_9^2 u_{ij} = \frac{1}{6h^2}\Big[u_{i-1,j-1} + 4u_{i,j-1} + u_{i+1,j-1} + 4u_{i-1,j} - 20u_{ij} + 4u_{i+1,j} + u_{i-1,j+1} + 4u_{i,j+1} + u_{i+1,j+1}\Big]$$

Expanding in Taylor's polyn.

$$= \nabla^2 u_{ij} + \frac{h^2}{12}\Big[(u_{4x})_{ij} + 2(u_{xxyy})_{ij} + (u_{4y})_{ij}\Big] + O(h^4).$$

Therefore, the local truncation error of this 9-point scheme to approximate the Laplace equ. is

$$\boxed{\begin{aligned} \tau_{ij} &= \nabla_9^2 u_{ij} - \nabla^2 u_{ij} \\ &= \frac{h^2}{12}\Big[(u_{4x})_{ij} + 2(u_{xxyy})_{ij} + (u_{4y})_{ij}\Big] + O(h^4). \end{aligned}} \tag{2.1}$$

Noticing that

$$\boxed{\nabla^2(\nabla^2 u) = \nabla^2(u_{xx} + u_{yy}) =}$$

$$(u_{xx})_{xx} + (u_{xx})_{yy} + (u_{yy})_{xx} + (u_{yy})_{yy}$$

$$= \boxed{u_{4x} + 2u_{xxyy} + u_{4y}} \qquad (3.1)$$

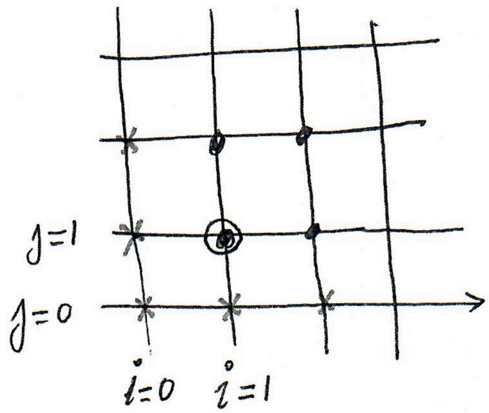
and $\nabla^2 u = 0$ if $u$ is a solution of (0.1)

then $T_{ij}$ in (2.1) reduces to

$$\boxed{T_{ij}} = \frac{h^2}{12}\big(\nabla^2(\underbrace{\nabla^2 u}_{=0})\big)_{ij} + \mathcal{O}(h^4) = \boxed{\mathcal{O}(h^4)}$$

As a consequence,

<u>$\nabla_9^2 U_{ij}$ scheme is an $\mathcal{O}(h^4)$ FDM for Laplace equation.</u>



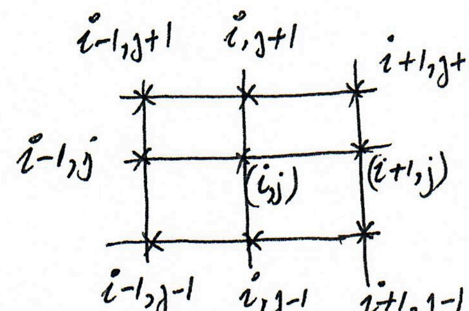

Bounday points

For $\underline{j=1}$, $\underline{i=1}$

$$\frac{1}{6h^2}\Big( U_{0,0} + 4U_{1,0} + U_{2,0} + 4U_{0,1} - 20U_{1,1} + 4U_{2,1} + U_{0,3} + 4U_{1,3} + U_{2,3} \Big)$$

B.P.

For <u>interior points</u> $(x_i, y_j)$.




$$\overbrace{1 \quad 4 \quad 1 \;\cdots\cdots\; 4}^{m+1 \text{ entries}} \quad \overbrace{-20 \quad 4 \;\cdots\cdots\; 1}^{m+1 \text{ entries}} \quad 4 \quad 1$$

$$1 \quad 4 \quad 1 \;\cdots\cdots\; 4 \quad -20 \quad 4 \;\cdots\cdots\; 1 \quad 4 \quad 1$$

$$S \qquad\qquad T \qquad\qquad S$$

Using <u>row-ordering</u> which is the same ordering that we used for 5-point stencil for Laplace, we arrive to

$$\begin{bmatrix} T & S & 0 & 0 & \cdots & 0 \\ S & T & S & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ & & S & T & & S \\ 0 & \cdots & & & S & T \end{bmatrix}_{[(m\times m)\times(m\times m)]}$$

where

$$T = \begin{bmatrix} -20 & 4 & 0 & 0 & \cdots & 0 \\ 4 & -20 & 4 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & & \\ \vdots & & & 4 & -20 & 4 \\ 0 & & & & 4 & -20 \end{bmatrix}_{m\times m}, \qquad S = \begin{bmatrix} 4 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 4 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & & \\ \vdots & & & 1 & 4 & 1 \\ 0 & & & & 1 & 4 \end{bmatrix}_{m\times m}$$

Now, consider Poisson's equation

$$\boxed{\nabla^2 u(x,y) = f(x,y)} \qquad (5.1)$$

we can easily define an $\mathcal{O}(h^4)$ finite difference method for (5.1). In fact,

<u>Theorem.</u> The finite difference scheme

$$\boxed{\tilde{\nabla}_9^2 U_{ij} = \nabla_9^2 U_{ij} - \frac{h^2}{12}(\nabla^2 f)_{i,j}} \qquad (5.2)$$

approximates (5.1) to $\mathcal{O}(h^4)$.

<u>Proof.</u>- (Homework problem)